CSC 665: Artificial Intelligence
Lecture: Logic IV

1 Choice of representation

So far we've been studying logical agents that operate in the language of propositional logic (or its restriction to Horn clauses). Since so much depends on our choice of language, it's worth asking whether we made the right choice. In other words, is propositional logic a good representational form?

As a thought experiment, let's compare propositional logic to English. Compared to propositional logic, English has the advantage of being much more expressive. Indeed, your college education has likely been communicated to you almost entirely in written and spoken English. It's hard to imagine doing the same thing in propositional logic.

On the other hand, English has a much more complicated syntax. Whereas we were able to write down a complete grammar for propositional logic in Backus-Naur form (BNF) using just 8 lines, no such grammar exists for the English language, and grammars that attempt to capture a large subset of English span much more than 8 lines.

Furthermore, the semantics of an English sentence are often ambiguous. For example, in the sentence, "scientists count whales from space," it is not clear whether the whales are from space, the scientists are from space, or the counting is being done from space. Additional context and grounding in the real world are often necessary to determine which interpretation is most likely to be correct, which is not the case for propositional logic.

In summary, although English is extremely powerful in terms of expressivity, it is too unwieldy to use as a representational foundation for logical agents. In the rest of this lecture, we explore first-order logic, which is a language that attempts to replicate some of the expressive features of English without sacrificing precision or compactness.

2 First-order logic by example

Our presentation of first-order logic will be much less formal than our study of propositional logic. We will focus on looking at examples rather than giving mathematically precise definitions as we examine the syntax and semantics of first-order logic.

2.1 Issues with propositional logic

Consider the English sentence, "Every CSC 665 student knows BFS." How would we encode this sentence in propositional logic? Well, given the course roster, we can define an atomic symbol for each student in the class and write the formula

$$\text{KentKnowsBFS} \wedge \text{JayKnowsBFS} \wedge \text{MichelleKnowsBFS} \wedge \dots,$$

where we have omitted the ~ 35 additional conjuncts required to encode the desired fact. This usage of propositional logic should feel fairly inelegant to you. If we try to pinpoint what exactly feels wrong, we can identify two issues.

1. **The representation is not compact.** It took a conjunction of 40 symbols to express an English sentence consisting of just six words. Furthermore, as the number of students in the class grows, the formula in propositional logic must also grow linearly, whereas the statement in English can remain unchanged. First-order logic addresses this problem by introducing *quantifiers* and *variables*.
2. **The representation is not factored.** We have to define a new propositional symbol for each student in the class. And since these symbols are atomic, there is nothing in the language that indicates that each symbol is expressing the same kind of relationship — that is, that a particular student knows about the algorithm BFS. In other words, propositional logic doesn't recognize that the world consists of objects with relations between them. First-order logic solves this problem by explicitly modeling *objects*, *predicates*, and *relations*.

Note that expressivity is more than just a matter of elegance. Indeed there are some statements that are impossible to encode using propositional logic. For example, the sentence, "every even integer larger than 2 is the sum of two primes" (Goldbach's conjecture), has no translation in propositional logic, since it would require the conjunction of an infinite number of atomic symbols.

2.2 Syntax of first-order logic

Here is a rough description of the syntax of first-order logic, with examples.

- A **term** is one of the following:
  - a **constant symbol** (BFS, Tyler)
  - a **variable** (x, y)
  - a **function** of terms (Hometown(Tyler), Sum$(x, 3)$)
- A **formula** has a truth value and is one of:
  - a **predicate** applied to terms (Knows(Tyler, BFS), read "Tyler knows BFS")
  - a **connective linking formulas** $(\text{665Student}(x) \implies \text{Knows}(x, \text{BFS}))$
  - a **quantifier** applied to a formula $(\forall x.\, \text{665Student}(x) \implies \text{Knows}(x, \text{BFS}))$
- A **quantifier** is one of:
  - $\forall$, the **universal quantifier** pronounced "for all"
  - $\exists$, the **existential quantifier** pronounced "there exists"

For completeness, here is the complete BNF grammar for first-order logic.

$$
\begin{array}{rcl}
\textit{Sentence} & \rightarrow & \textit{AtomicSentence} \mid \textit{ComplexSentence} \\
\textit{AtomicSentence} & \rightarrow & \textit{Predicate} \mid \textit{Predicate}(\textit{Term}, \ldots) \mid \textit{Term} = \textit{Term} \\
\textit{ComplexSentence} & \rightarrow & (\ \textit{Sentence}\ ) \\
 & \mid & \neg\ \textit{Sentence} \\
 & \mid & \textit{Sentence} \wedge \textit{Sentence} \\
 & \mid & \textit{Sentence} \vee \textit{Sentence} \\
 & \mid & \textit{Sentence} \ \Rightarrow\ \textit{Sentence} \\
 & \mid & \textit{Sentence} \ \Leftrightarrow\ \textit{Sentence} \\
 & \mid & \textit{Quantifier}\ \textit{Variable}, \ldots\ \textit{Sentence} \\
 & & \\
\textit{Term} & \rightarrow & \textit{Function}(\textit{Term}, \ldots) \\
 & \mid & \textit{Constant} \\
 & \mid & \textit{Variable} \\
 & & \\
\textit{Quantifier} & \rightarrow & \forall \mid \exists \\
\textit{Constant} & \rightarrow & A \mid X_1 \mid \textit{John} \mid \cdots \\
\textit{Variable} & \rightarrow & a \mid x \mid s \mid \cdots \\
\textit{Predicate} & \rightarrow & \textit{True} \mid \textit{False} \mid \textit{After} \mid \textit{Loves} \mid \textit{Raining} \mid \cdots \\
\textit{Function} & \rightarrow & \textit{Mother} \mid \textit{LeftLeg} \mid \cdots
\end{array}
$$

2.3 Semantics of first-order logic

Although it is possible to present the semantics of first-order logic in terms of models that either satisfy or do not satisfy a given formula (as we did for propositional logic), it is more instructive to study examples of formulas in first-order logic along with their "translations" in English.

- *Tyler is not a student.* We introduce a Student predicate, along with a constant symbol Tyler, allowing us to write
$$\neg\text{Student}(\text{Tyler}).$$
- *Tyler teaches at SF State.* We introduce a TeachesAt predicate that takes in two arguments, as well as the constant symbol SFSU:
$$\text{TeachesAt}(\text{Tyler}, \text{SFSU}).$$
- *Anyone who teaches at SF State does not teach at San Jose State.* We introduce another constant symbol SJSU, then write the statement using a universal quantifier along with some familiar connectives from propositional logic:
$$\forall x.\, \text{TeachesAt}(x, \text{SFSU}) \implies \neg\text{TeachesAt}(x, \text{SJSU}).$$
- *Tyler teaches at some university.* This one requires the existential quantifier, along with one additional predicate to indicate whether an object is a university:
$$\exists x.\, \text{University}(x) \wedge \text{TeachesAt}(\text{Tyler}, x).$$
- *Every teacher teaches at a university.* This example shows that the two quantifiers can be combined:
$$\forall x.\, \text{Teacher}(x) \implies (\exists y.\, \text{University}(y) \wedge \text{TeachesAt}(x, y)).$$
- *Every teacher teaches at the same university.* This absurd claim shows that the order of quantifiers makes a big difference in the meaning of a formula:
$$\exists y.\, \text{University}(y) \wedge (\forall x.\, \text{Teacher}(x) \implies \text{TeachesAt}(x, y)).$$

Note that implication ($\implies$) is almost always paired with the universal quantifier ($\forall$), and conjunction ($\wedge$) is almost always paired with the existential quantifier ($\exists$).

The following sentences in English all have at least two possible interpretations. Which meaning (as given unambiguously in first-order logic) do you think was intended in each case?

- *Everybody loves somebody.*
$$\exists y.\, \forall x.\, \text{Loves}(x, y)$$
or
$$\forall x.\, \exists y.\, \text{Loves}(x, y)$$
- *One American dies of melanoma every hour.*
$$\exists x.\, \text{American}(x) \wedge \forall h.\, \text{Hour}(h) \implies \text{DiesAt}(x, h)$$
or
$$\forall h.\, \text{Hour}(h) \implies \exists x.\, \text{American}(x) \wedge \text{DiesAt}(x, h)$$
- *Everyone is either tall or short.*
$$(\forall x.\, \text{Tall}(x)) \oplus (\forall x.\, \text{Short}(x))$$
or
$$\forall x.\, \text{Tall}(x) \oplus \text{Short}(x)$$